On the Zeros of Entire Almost Periodic Functions

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We shal prove in this paper that a lattice $\Omega \subset \mathbf{C}$ which is the set of zeros of an entire almost periodic function f: $\mathbf{C} \to \mathbf{C}$ is periodic in the direction of almost periodicity.

A method for construction of holomorphic almost periodic functions was introduced in [4] and was applied more extensively in [6]. During the work on [6] the authors noticed that the method failed if the set of zeros was a lattice and not periodic in the direction of almost periodicity. The authors discussed it only briefly and it was not mentioned in the paper.

A rotation of both the lattice and the direction of almost periodicity around the point 0 and by the same angle will have no influence on the existence of almost periodic entire functions with the given lattice as set of zeros.

Accordingly, we shall assume that the given direction of almost periodicity is the direction of the real axis and that the lattice Ω is not periodic in this direction, i.e. that 0 is the only real number in Ω. We shall study a hypothetical entire almost periodic function f: $\mathbf{C} \to \mathbf{C}$ with $f^{-1}(0) = \Omega$. We are going to prove the non existence of such a function by deducing that some function derived from f will have properties which contradict each other.

The 8 lemmas of this paper are statements directly or indirectly dealing with the non existing function f. Hence, they have no applications whatever beyond the scope of this paper. The 7 propositions are genuine statements about rather general classes of functions, but most of them are reformulations of known results adopted for our particular purpose.

The first section states the problem, introduces some notions and does some preliminary work. It ends with the key lemmas 2 and 3, which state that f and some related functions cannot assume very small values except near the zeros.

The second section investigates the Fourier series of f. It turns out that the 2-dimensionality of the lattice of zeros is reflected in the set of Fourier exponents. In fact the subspace of the $\mathbf{Q}$-vector space $\mathbf{R}$ generated by the set of Fourier exponents has a 'compulsory' 2-dimensional subspace determined by Ω.

In the third section we introduce the spatial extension of f, i.e. a function F: $\mathbf{R}^\infty \times \mathbf{R} \to \mathbf{C}$ with $f(z) = F(\gamma x; y)$; $z = x + iy$. Here, F is limit periodic and $\gamma = (\gamma_1, \gamma_2, \ldots)$ is a base for the vector space generated by the Fourier exponents such that (γ_1, γ_2) spans the compulsory subspace. If M denotes the zeros of F in the $(x_1, x_2; y)$-subspace of $\mathbf{R}^\infty \times \mathbf{R}$, we have $F^{-1}(0) = p^{-1}(M)$ when p: $\mathbf{R}^\infty \times \mathbf{R} \to \mathbf{R}^2 \times \mathbf{R}$ is the projection. Further, M is a system of parallel straight lines, each connecting a point of Ω placed in $\mathbf{R}^\infty \times \mathbf{R}$ by $f(z) = F(\gamma x; y)$ and projected on the subspace, with a point of the unit lattice in the

(x_1, x_2)-plane. The proof of this is the tiresome part of the paper and the author hopes that somebody will find a more elegant way of doing it.

The fourth section finishes the proof of the non existence of f by a topological argument. We know that f has the variation of its argument around each zero equal to 2π. It is possible to let small circles around the zeros of f slide along the lines of M to end in the (x_1, x_2)-plane and this enables us to prove that also the restriction $\varphi(x_1, x_2) = F(x_1, x_2, 0, 0, ..., 0)$ by convenient orientation of the (x_1, x_2)-plane has the variation of the argument around each zero equal to 2π. The lemmas 2 and 3 will also carry over and that makes it possible to prove that the variation of the argument of F along the boundaries of certain big squares has to be zero and also to be a very large number and that is the contradiction.

In the fifth and last section we shall prove that there is a lattice $\Omega' \subset \mathbf{C}$ and a second order entire almost periodic function with $\Omega \cup \Omega'$ as its set of zeros.

Almost periodic properties of the function f

The field $\mathbf{R}$ of real numbers is also a $\mathbf{Q}$-vector space and we shall use the notion of linear independence accordingly. If (x_j) is a sequence of real numbers which are linearly independent over $\mathbf{Q}$ we shall simply say that the numbers x_j, the sequence (x_j) or the set $\underline{x} = (x_1, x_2, ...)$ are independent.

We shall assume that the lattice Ω is spanned by the complex numbers $\omega_1 = \alpha_1 + i\beta_1$, $\omega_2 = \alpha_2 + i\beta_2$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbf{R}$, and that the indices are chosen such that $\alpha_1\beta_2 - \alpha_2\beta_1 = \Delta > 0$. We shall also assume that Ω is not periodic in the direction of the real axis, and this is equivalent to the assumption that $\Omega \cap \mathbf{R} = \{0\}$ and also to the assumption that β_1 and β_2 are independent.

We shall call a set $T \subseteq \mathbf{C}$ relatively dense if there exists a real number L, such that every closed interval $I \subset \mathbf{R}$ of length L contains the real part of at least one element of T. By Kronecker's theorem and the Bohl-Wennberg theorem ([6] p. 145, footnote) the following statement holds:

For every $\delta > 0$ and every $y \in \mathbf{R}$ the set of numbers $\omega \in \Omega$ with imaginary part in the interval $[y - \delta, y + \delta]$ is relatively dense.

We shall consistently use z with or without indices as notation for a complex number, and always with $z = x + iy$ and the indices repeated on x and y. To a bounded interval $I \subset \mathbf{R}$ corresponds a strip $S_I = \{z \mid y \in I\}$. A strip $S \subset \mathbf{C}$ is a set defined in this way. We shall write I_S for the interval defining S. Mostly, we have $I = [-A, A]$ with some $A > 0$ and we shall then write S_A for S_I.

Most of the functions considered here will be continuous functions g: $\mathbf{C} \to \mathbf{C}$, but not always holomorphic. We define the absolute value $|g|$ by $|g|(z) = |g(z)|$. We shall permit ourselves the abuse of notation of confusing a function with its value, e.g. by writing "the function $g(z)e^{\gamma z}$" meaning "the function $\tilde{g}$: $\mathbf{C} \to \mathbf{C}$ defined by $\tilde{g}(z) = g(z)e^{\gamma z}$". For $\tau \in \mathbf{C}$ we use the notation g_τ: $\mathbf{C} \to \mathbf{C}$ for the translated function $g_\tau(z) = g(z+\tau)$. For $\varepsilon > 0$, $A > 0$ we call $\tau \in \mathbf{C}$ an (ε, A)-translation number of g if $|g_\tau(z) - g(z)| \leqq \varepsilon$ for every $z \in S_A$. According to Bohr's definition g is almost periodic if g is continuous and the set of real (ε, A)-translation numbers is relatively dense for every $\varepsilon > 0$, $A > 0$. This definition is equivalent to Bochner's definition, according to which g: $\mathbf{C} \to \mathbf{C}$ is almost periodic if g is continuous and every sequence $(\tau_j \mid j \in \mathbf{N})$ of real numbers has a subsequence (τ'_j) such that the sequence $(g_{\tau'_j})$ converges uniformly in every strip. This can be generalized in the following way:

PROPOSITION 1. *Let g:* $\mathbf{C} \to \mathbf{C}$ *be almost periodic and let S be a strip. Then every sequence* (τ_j) *of complex numbers* $\tau_j \in S$ *has a subsequence* (τ'_j) *such that* $(g_{\tau'_j})$ *converges uniformly in every strip.*

PROOF: We write $\tau_j = \rho_j + i\sigma_j$ and we can then choose the subsequence (τ'_j) with $\tau'_j = \rho'_j + i\sigma'_j$ such that $(g_{\rho'_j})$ converges uniformly in every strip and (σ'_j) converges to a limit σ. Then, obviously $(g_{\rho'_j+i\sigma})$ converges uniformly in every strip, and since g is uniformly continuous in every strip, the sequence $(g_{\tau'_j} - g_{\rho'_j+i\sigma})$ tends to 0 uniformly in every strip, and the statement follows.

We shall use the following statement, which is pure function theory and not very exciting:

PROPOSITION 2. *Let* $\mathscr{B}$ *denote the* $\mathbf{C}$*-vector space of entire functions bounded in every strip with the Fréchét-space topology corresponding to uniform convergence in every strip. Let* $\mathscr{A} \subset \mathscr{B}$ *be the subset of functions g:* $\mathbf{C} \to \mathbf{C}$ *with* $g^{-1}(0)$ *equal to* Ω *or* $\mathbf{C}$. *Then* $\mathscr{A}$ *is a closed subset of* $\mathscr{B}$.

PROOF: We shal prove that $\mathscr{B}\backslash\mathscr{A}$ is open. That $h \in \mathscr{B}\backslash\mathscr{A}$ means that h: $\mathbf{C} \to \mathbf{C}$ is entire and that there is either a number $\omega \in \Omega$ with $h(\omega) \neq 0$ or a number $z_0 \in \mathbf{C}\backslash\Omega$ with $h(z_0) = 0$. In the first case it is obvious that h is in the interior of $\mathscr{B}\backslash\mathscr{A}$. In the second case there is a disc $D \subset \mathbf{C}$ with center z_0 and positive distance from Ω, and then $|h|$ has infimum $k > 0$ on the boundary of D and according to Rouché's theorem every entire function approximating h with accuracy better than k on the boundary of D has a zero in D and that proves again that h is in the interior of $\mathscr{B}\backslash\mathscr{A}$. That finishes the proof.

We shall start in earnest on our non existence proof. From now on f: $\mathbf{C} \to \mathbf{C}$ is an entire function which is also almost periodic and satisfies that $f^{-1}(0) = \Omega$. Until the end of section 4 we shall use f exclusively as notation for this particular function.

LEMMA 1. *To* $\varepsilon > 0$, $\delta > 0$, $A > 0$ *corresponds* $\varepsilon' > 0$ *such that every* $(\varepsilon', A + \delta)$*-translation number* τ *of f has a corresponding* (ε, A)*-translation number* $\omega \in \Omega$ *of f with* $|\tau - \omega| \leqq \delta$.

PROOF: Let P denote the closed parallelogram with corners $\pm\frac{1}{2}\omega_1 \pm \frac{1}{2}\omega_2$ and $P(\tau)$, $\tau \geqq 0$ the set of numbers $z \in P$ with $|z| \geqq \tau$. With $\tau_0 = \frac{1}{2}\min\{|\omega| \mid \omega \in \Omega \setminus \{0\}\}$ we define $\kappa: [0,\tau_0] \to [0,\infty[$ by $\kappa(\tau) = \inf|f|(P(\tau))$. We choose $\tau_1 > 0$ such that $\tau_1 \leqq \delta$, $\tau_1 \leqq \tau_0$ and $|f(z_0) - f(z_1)| \leqq \frac{\varepsilon}{2}$ for $z_1, z_2 \in S_{A+\delta}$, $|z_2 - z_1| \leqq \tau_1$. Next, we choose $\varepsilon' \in]0, \frac{\varepsilon}{2}[$ such that $\varepsilon' < \kappa(\tau_1)$. For the given $(\varepsilon', A+\delta)$-translation number τ of f we choose $\omega \in \Omega$ such that $\tau - \omega \in P$. Since $f(-\omega) = 0$, we have $|f(\tau - \omega)| \leqq \varepsilon'$, hence $\tau - \omega \notin P(\tau_1)$, but that implies that $|\tau - \omega| \leqq \tau_1 < \delta$. For $z \in S_A$ we have $z - (\tau - \omega) \in S_{A+\delta}$ and we get the estimate

$$
\begin{aligned}
&|f(z+\omega) - f(z)| \\
&\leqq |f(z+\omega) - f(z+\omega-\tau)| + |f(z-(\tau-\omega)) - f(z)| \leqq \varepsilon' + \frac{\varepsilon}{2} \leqq \varepsilon,
\end{aligned}
$$

which proves the lemma.

LEMMA 2. *For $A > 0$ we define $S_A(r)$ as the subset of points of S_A with distance $\geqq r$ from Ω* and we define $k_A : [0,r_0] \to [0,\infty[$ *by $k_A(r) = \inf|f|(S_A(r))$ with r_0 as in the proof of Lemma 1. Then k_A is strictly positive on* $]0,r_0]$.

PROOF: We do it indirectly assuming that $k_A(r_1) = 0$ for some $r_1 \in]0,r_0]$. Then there is a sequence (z_j) with $z_j \in S_A(r_1)$ and $(f(z_j)) \to 0$. Let P be the parallelogram from the proof of Lemma 1. We choose (ω_j) with $\omega_j \in \Omega$ such that $z_j - \omega_j \in P$, $j \in \mathbf{N}$. By replacing (z_j) by a convenient subsequence (which we shall still denote (z_j)) we can according to Proposition 1 assume that (f_{ω_j}) converges uniformly in every strip to an entire function $\tilde{f}: \mathbf{C} \to \mathbf{C}$, and by the compactness of P we can further assume that $(z_j - \omega_j) \to a \in P$. Since $(z_j - (a+\omega_j)) \to 0$ and f is uniformly continuous in every strip, we have also $(f(a+\omega_j)) \to 0$, but $(f(a+\omega_j)) = (f_{\omega_j}(a)) \to \tilde{f}(a)$, hence $\tilde{f}(a) = 0$. But $z_j \in S_A(r_1)$ implies that $z_j - \omega_j \in P(\mathrm{r}_1)$, and we have $f_{\omega_j} \in \mathscr{A}$, $j \in \mathbf{N}$ and Proposition 2 yields that $\tilde{f} \in \mathscr{A}$, hence $\tilde{f}^{-1}(0) = \Omega$ in contradiction to $\tilde{f}(a) = 0$. That proves the lemma.

LEMMA 3. *With P as in the proof of Lemma 1 and* $b = \max\{|y| \mid z \in P\}$ *we define T_f as the closure in $\mathscr{B}$ of $\{f_\tau \mid \tau \in \mathbf{R}\}$. For every $\tilde{f} \in T_f$ there is then an $a \in P$ such that $\tilde{f}_a^{-1}(0) = \Omega$. Further, for $A > 0$ and k_A as in Lemma 2 we have $|\tilde{f}(z)| \geqq k_{A+b}(r)$* for every *$z \in S_A$ with distance $\geqq r$ from every zero of $\tilde{f}$.*

PROOF: There is a sequence (τ_j) of real numbers such that $(f_{\tau_j}) \to \tilde{f}$ uniformly in every strip. We choose (ω_j) with $\omega_j \in \Omega$ and $\tau_j - \omega_j \in P$. By replacing (τ_j) by a convenient subsequence we can assume that $(\tau_j - \omega_j) \to -a$, and since P is symmetric, we have $a \in P$. Since $(\tau_j - (\omega_j - a)) \to 0$ and f is uniformly continuous in every strip, we have $(f_{\omega_j - a}) \to \tilde{f}$ and $(f_{\omega_j}) \to \tilde{f}_a$ uniformly in every strip. Since $f_{\omega_j} \in \mathscr{A}$, the first statement in the lemma follows from Proposition 2. By Lemma 2 it is quite obvious that $|f_{\omega_j}(z)| \geqq k_{A+b}(r)$ for every $z \in S_A$, and the last statement follows by passage to the limit. This ends the proof.

The Fourier exponents of f

Let $g: \mathbf{C} \to \mathbf{C}$ be an arbitrary entire almost periodic function. The function $a: \mathbf{R} \to \mathbf{C}$ defined by
$a(\lambda) = \lim_{T\to\infty} \frac{1}{T}\int_0^T g(x+iy)e^{-2\pi i\lambda(x+iy)}\,dx$ satisfies that $\Lambda_g = \{\lambda \in \mathbf{R} \mid a(\lambda) \neq 0\}$ is at most denumerable so that we have a Fourier series $\Sigma_{\lambda\in\Lambda_g} a(\lambda)e^{2\pi i\lambda z}$. The main theorem in the theory of almost periodic functions states that the Fourier series is summable with sum $g(z)$ and uniformly in every strip. In a more precise form this means that there is a function $k: \Lambda_g \times \mathbf{N} \to [0,1]$ with the following 3 properties:

(1) The set $\{\lambda \in \Lambda_g \mid k(\lambda, n) \neq 0\}$ is finite for every $n \in \mathbf{N}$.
(2) The sequence $(k(\lambda, n))$ tends to 1 for every fixed $\lambda \in \Lambda_g$.
(3) The sequence (s_n) of finite sums
$s_n(z) = \Sigma_{\lambda\in\Lambda_g} k(\lambda, n)a(\lambda)e^{2\pi i\lambda z}$ tends to $g(z)$ uniformly in every strip.

The vector space $\bar{\Lambda}_g \subset \mathbf{R}$ spanned by Λ_g has a base $\gamma = (\gamma_1, \gamma_2, ...)$. It may be a finite base $\gamma = (\gamma_1, ..., \gamma_m)$, but we shall formulate the following investigations as if the worst happens and only occasionally refer to the rather obvious changes to be made if the basis is finite. By the way, it is easy to see that there is an entire almost periodic function $h: \mathbf{C} \to \mathbf{C}$ such that $g(z)e^{h(z)}$ has the base infinite.

It is very important for our investigations that there is a fundamental relationship between the translation numbers of g and the base γ. This is described in detail in [1] where it is used in the proof of the approximation theorem, and the main points are summarized in [6] p 144-145 and 149-150. Unfortunately, the results are not formulated in terms of the base. We shall reformulate them and add a few remarks in way of proving them.

In this connection we must consider some Diophanthine inequalities of the form $|\gamma\tau - c| \leqq \delta \ (\mathrm{mod}\ n!\mathbf{Z})$ with $\delta > 0$; $\gamma, c \in \mathbf{R}$, $n \in \mathbf{N}$. That $\tau \in \mathbf{R}$ is solution of the inequality means that there is a $\nu \in \mathbf{Z}$ such that $|\gamma\tau - c - n!\nu| \leqq \delta$. In connection with the base γ we consider the following system of simultaneous Diophanthine inequalities where the second line gives the alternative form for $\gamma = (\gamma_1, ..., \gamma_m)$

$$|\gamma_j\tau| \leqq \delta\ (\mathrm{mod}\ n!\mathbf{Z}),\ j = 1, ..., n \tag{1}$$
$$|\gamma_j\tau| \leqq \delta\ (\mathrm{mod}\ n!\mathbf{Z}),\ j = 1, ..., m.$$

The relationship between γ and the translation numbers of g is given by the proposition:

PROPOSITION 3. *To $\varepsilon > 0$, $A > 0$ correspond $\delta > 0$, $n \in \mathbf{N}$ such that every solution of (1) is an (ε, A)-translation number of g.*

In fact, $\tau \in \mathbf{R}$ is an (ε, A)-translation number of g if it is an $(\frac{\varepsilon}{2}, A)$-translation number of the finite sum s_n which approximates g in S_A with accuracy $\frac{\varepsilon}{4}$. If q is the number of terms in s_n, it follows that τ is an (ε, A)-translation number τ of g, if it is an $(\frac{\varepsilon}{2q}$, $A)$-translation number of each term $k(\lambda, n)\ a\ (\lambda)e^{2\pi i\lambda z}$, and this will happen, if τ satisfies a set of Diophanthine inequalities $|\lambda_j\tau| \leqq \delta'$ (mod $\mathbf{Z}$), $j = 1, \ldots, q$. We express the λ_j in terms of the γ_j and choose n large enough such that the denominators in the coefficients in these expression are divisors in $n!$ and the proposition follows easily.

There is also a reverse relationship:

PROPOSITION 4. *If $\lambda \in \mathbf{R}$ has the property that to every $A > 0$, $\delta > 0$ exists an $\varepsilon > 0$ such that every (ε, A)-translation number τ og g satisfies the Diophanthine inequality $|\lambda\tau| \leqq \delta$ (mod $\mathbf{Z}$), then $\lambda \in \bar{\Lambda}_g$.*

PROOF: If $\lambda \notin \bar{\Lambda}_g$, the numbers $\lambda, \gamma_1, \gamma_2, \ldots$ are independent and Kronecker's theorem tells us that for every $\delta > 0$, $n \in \mathbf{N}$ some solutions of (1) also satisfies the inequality $|\lambda\tau - \frac{1}{2}| \leqq \delta$ (mod $n!\mathbf{Z}$). Hence, for $\delta < \frac{1}{4}$ the condition in the proposition is not satisfied by λ. This proves the proposition.

We shall now return to the hypothetical function f, but first some formulas concerning Ω must be established. For $\omega = \alpha + i\beta \in \Omega$ with $\omega = n_1\omega_1 + n_2\omega_2$, $n_1, n_2 \in \mathbf{Z}$ we have

$$\alpha = n_1\alpha_1 + n_2\alpha_2, \quad \beta = n_1\beta_1 + n_2\beta_2.$$

Eliminating either n_2 or n_1 between these, we get the 2 sets of relations

$$n_2 = -n_1\frac{\beta_1}{\beta_2} + \frac{\beta}{\beta_2}; \; \alpha = n_1\frac{\Delta}{\beta_2} + \frac{\alpha_2}{\beta_2}\beta; \; \omega = n_1\frac{\Delta}{\beta_2} + \frac{\omega_2}{\beta_2}\beta. \tag{2}$$

$$n_1 = -n_2\frac{\beta_2}{\beta_1} + \frac{\beta}{\beta_1}; \; \alpha = -n_2\frac{\Delta}{\beta_1} + \frac{\alpha_1}{\beta_1}\beta; \; \omega = -n_2\frac{\Delta}{\beta_1} + \frac{\omega_1}{\beta_1}\beta. \tag{3}$$

For the function f we shall simply use Λ as notation for the set of Fourier exponents and $\bar{\Lambda}$ for the vector space spanned by Λ. The following lemma tells that $\bar{\Lambda}$ is at least 2-dimensional and, hence, f is not limit periodic.

LEMMA 4. $\frac{\beta_1}{\Delta}, \frac{\beta_2}{\Delta} \in \bar{\Lambda}$.

PROOF: For $\delta > 0$ we choose $\delta_1 = \frac{\Delta\delta}{|\beta_1|+|\omega_1|}$ and for $A > 0$ we can by Lemma 1 choose $\varepsilon > 0$ such that every $(\varepsilon, A+\delta)$-translation number of τ of f has a corresponding $\omega = \alpha + i\beta = n_1\omega_1 + n_2\omega_2 \in \Omega$ with $|\tau - \omega| \leqq \delta_1$.

In particular, if $\tau \in \mathbf{R}$ we get $|\beta| \leqq \delta_1$ and (3) yields

$$|\tau + n_2 \frac{\Delta}{\beta_1}| \leqq \delta_1 + \frac{|\omega_1|}{|\beta_1|}|\beta| \leqq \left(1 + \frac{|\omega_1|}{|\beta_1|}\right)\delta_1 = \frac{\Delta}{|\beta_1|}\delta,$$

hence $|\tau \cdot \frac{\beta_1}{\Delta} + n_2| \leqq \delta$, which is exactly $\left|\frac{\beta_1}{\Delta}\tau\right| \leqq \delta$ (mod $\mathbf{Z}$). Thus, it follows from Proposition 4 that $\frac{\beta_1}{\Delta} \in \bar{\Lambda}$, and that $\frac{\beta_2}{\Delta} \in \bar{\Lambda}$ is proved in the same way.

Since $\frac{\beta_1}{\Delta}, \frac{\beta_2}{\Delta}$ are independent, we can choose the base $\underline{\gamma}$ with $\gamma_1 = \frac{\beta_1}{\Delta}, \gamma_2 = \frac{\beta_2}{\Delta}$ and from now on we shall assume that $\underline{\gamma}$ is chosen like that, and the subspace of $\bar{\Lambda}$ spanned by γ_1 and γ_2 will be called the compulsory subspace.

The spatial extension of f

We shall introduce some functions defined on spaces of pairs $(\underline{x}; y)$ of a finite or infinite sequence $\underline{x} = (x_1, \dots, x_m)$ or $\underline{x} = (x_1, x_2, \dots)$ of real numbers, and a real number y. We shall denote the spaces $\mathbf{R}^m \times \mathbf{R}$ or $\mathbf{R}^\infty \times \mathbf{R}$ accordingly and they shall always be organized with the product topology. We shall formulate everything for $\mathbf{R}^\infty \times \mathbf{R}$ only.

If $I \subset \mathbf{R}$ is a bounded interval, we shall call the set $Sl_I = \{(\underline{x}; y) \in \mathbf{R}^\infty \times \mathbf{R} \mid y \in I\}$ the slice corresponding to I, and a slice shall be a set defined in this way by some bounded interval. If $I = [-A, A]$, we shall also write Sl_A for Sl_I. A function $G : \mathbf{R}^\infty \times \mathbf{R} \to \mathbf{C}$ is called limit periodic if it is continuous and satisfies the following condition: To $\varepsilon > 0$, $A > 0$ corresponds $n \in \mathbf{N}$ such that $|G(\underline{x}''; y) - G(\underline{x}'; y)| \leqq \varepsilon$ if $|y| \leqq A$ and $x_1'' - x_1', \dots x_n'' - x_n'$ are integers divisible by $n!$. It is easy to prove that G is limit periodic if and only if it can be approximated uniformly in any given slice with any given accuracy by a continuous function depending only on finitely many variables $x_1, \dots, x_m; y$ and with an integral period in $x_1, \dots, x_m$. However, we shall not use that.

PROPOSITION 5. *Let* $G : \mathbf{R}^\infty \times \mathbf{R} \to \mathbf{C}$ *be limit periodic and* $\underline{\gamma} = (\gamma_1, \gamma_2, \dots)$ *independent. We define* $g : \mathbf{C} \to \mathbf{C}$ *by* $g(z) = G(\underline{\gamma}x; y)$. *Then g is almost periodic and* Λ_g *is contained in the vector space spanned by* $\gamma_1, \gamma_2, \dots$.

PROOF: Let $\varepsilon > 0, A > 0$ be given. We choose $\delta > 0, n \in \mathbf{N}$ such that $|G(\underline{x}'';y) - G(\underline{x}';y)| \leqq \frac{\varepsilon}{2}$ if either $|x_j'' - x_j'| \leqq \delta, j = 1,\ldots, n$; $|y| \leqq A$ or $x_j'' - x_j'$ for $j = 1,\ldots, n$ is an integer divisible by $n!$ and $|y| \leqq A$. Let τ be a real solution of the inequalities (1). We can choose integers $\nu_1,\ldots, \nu_m$ such that for every $x \in \mathbf{R}$ we have $|\gamma_j(x+\tau) - (\gamma_j x+n!\nu_j)| \leqq \delta, j = 1,\ldots, n$. With $\underline{x}' = (\gamma_1 x + n!\nu_1,\ldots, \gamma_n x + n!\nu_n, \gamma_{n+1}x, \gamma_{n+2}x,\ldots)$ we have the inequalities

$$|g(z + \tau) - G(\underline{x}' ;y)| \leqq \frac{\varepsilon}{2}\ ;\quad |G(\underline{x}';y) - g(z)| \leqq \frac{\varepsilon}{2},$$

which prove that τ is an (ε, A)-translation number of g. Since the set of real solutions of (1) is relatively dense, this proves that g is almost periodic.

Let λ be a Fourier exponent of g and τ an (ε, A)-translation number of g for some $\varepsilon > 0$, $A > 0$. Then we have

$$a(\lambda)\ (e^{2\pi i\lambda\tau} - 1) = \lim_{T\to\infty} \frac{1}{\mathrm{T}} \int_0^T (g(x + \tau) - g(x))e^{-2\pi i\lambda x}dx,$$

which yields the estimate

$$|a(\lambda)|\ |e^{2\pi i\lambda\tau} - 1| \leqq \varepsilon.$$

On the other hand, if $|\lambda\tau - \frac{1}{2}| \leqq \frac{1}{4}$ (mod $\mathbf{Z}$) we obviously have

$$|a(\lambda)|\ |e^{2\pi i\lambda\tau} - 1| \geqq |a(\lambda)|.$$

If λ is not in the vector space spanned by $\gamma_1, \gamma_2,\ldots$, some solutions of (1) will by Kronecker's theorem also satisfy that $|\lambda\tau - \frac{1}{2}| \leqq \frac{1}{4}$ (mod $\mathbf{Z}$), so that they cannot be (ε, A)-translation numbers of g for any $A > 0$ and any $\varepsilon < |a(\gamma)|$. Thus λ is in the space spanned by $\gamma_1\ \gamma_2,\ldots$, and that ends the proof.

With G and g as in Proposition 5 we shall call g the diagonal function of G corresponding to $\underline{\gamma}$ and G a spatial extension of g corresponding to the base $\underline{\gamma}$ of Λ_g. The subspace $C = \{(\underline{\gamma}x;y) \mid x, y \in \mathbf{R}\}$ will be called the $\underline{\gamma}$-diagonal in $\mathbf{R}^\infty \times \mathbf{R}$ and the affine subspaces $C_{\underline{x}} = \{(\underline{x} + \underline{\gamma}x;y) \mid x, y \in \mathbf{R}\}, \underline{x} \in \mathbf{R}^\infty$ will be called the analytic planes in $\mathbf{R}^\infty \times \mathbf{R}$.

PROPOSITION 6. *Let $g : \mathbf{C} \to \mathbf{C}$ be entire and almost periodic, and let $\underline{\gamma} = (\gamma_1, \gamma_2,\ldots)$ be a base for $\bar{\Lambda}_g$. Then g has a uniquely determined spatial extension $G : \mathbf{R}^\infty \times \mathbf{R} \to \mathbf{C}$, and for every $\underline{x} \in \mathbf{R}^\infty$ the function $g_{\underline{x}} : \mathbf{C} \to \mathbf{C}$ defined by $g_{\underline{x}}(z) = G(\underline{x}+\underline{\gamma}x; y)$ is entire and almost periodic and belongs to the closure T_g of $\{g_\tau \mid \tau \in \mathbf{R}\} \subset \mathscr{B}$.*

PROOF: With $X_n = \mathbf{R} \times \mathbf{Z}^n$; $n \in \mathbf{N}$ we define $\psi_n : X_n \to \mathbf{R}^\infty$ by

$$\psi_n(x; \nu_1,\dots, \nu_n) = (\gamma_1 x + n!\nu_1,\dots, \gamma_n x + n!\nu_n, \gamma_{n+1}x, \gamma_{n+2}x,\dots).$$

For each $n \in \mathbf{N}$ the set $M_n = \psi_n(X_n) \times \mathbf{R} \subset \mathbf{R}^\infty \times \mathbf{R}$ is a system of analytic planes. Since ψ_n is injective, we can define $G_n : M_n \to \mathbf{C}$ by

$$G_n(\psi_n(x; \nu_1,\dots, \nu_n); y) = g(x + iy) = g(z).$$

For every $\underline{x} \in \psi_n(X_n)$ we have $\underline{x} + \underline{\gamma}x \in \psi_n(X_n)$ for every $x \in \mathbf{R}$ so that we can define $g_{n,\underline{x}} : C_{\underline{x}} \to \mathbf{C}$ by $g_{n,\underline{x}}(z) = G_n(\underline{x} + \underline{\gamma}x; y)$. Further, there is a $\tau \in \mathbf{R}$ and $\nu_1,\dots, \nu_n \in \mathbf{Z}$ such that $\underline{x} = \psi_n(\tau; \nu_1,\dots, \nu_n)$, hence $\underline{x} + \underline{\gamma}x = \psi_n(x + \tau; \nu_1,\dots, \nu_n)$ and $g_{n,\underline{x}} = g_\tau$. We have thus $g_{n,\underline{x}} \in T_g$.

Let us consider an arbitrary $\underline{x}^\circ \in \mathbf{R}^\infty$ with its corresponding analytic plane $C_{\underline{x}^\circ}$. For $n \in \mathbf{N}$ we define

$$U_n(\underline{x}^\circ) = \{\underline{x} \in \mathbf{R}^\infty \mid |x_j - x_j^\circ| \leqq \frac{1}{2n}, j = 1,\dots, n\},$$

and the $U_n(\underline{x}^\circ)$ constitute a base for the neighbourhoods of x° in $\mathbf{R}^\infty$. For $q, n \in \mathbf{N}$ we have $\psi_n(X_n) \cap U_q(\underline{x}^\circ) \neq \emptyset$ if the Diophanthine inequalities

$$|\gamma_j x - x_j^\circ| \leqq \frac{1}{2q} \ (\mathrm{mod}\ n!\mathbf{Z}), \quad j = 1,\dots, q \tag{4}$$

are satisfied by some $x \in \mathbf{R}$. By Kronecker's theorem this is always the case. We have thus proved that the sets $\psi_n(X_n)$, $n \in \mathbf{N}$ are dense in $\mathbf{R}^\infty$. We are interested in the analytic plane $C_{\underline{x}^\circ}$, and by its nth set of neighbour planes we mean the set $V_n(\underline{x}^\circ)$ of planes $C_{\underline{x}}$ with $\underline{x} \in \psi_n(X_n) \cap U_n(\underline{x}^\circ)$. Similarly the set $W_n(\underline{x}^\circ)$ of corresponding entire functions $g_{n,\underline{x}}$ is called the nth set of neighbour functions of $C_{\underline{x}^\circ}$.

For $\varphi \in \mathscr{B}$ and $A > 0$ we define the norm $\|\varphi\|_A = \sup |\varphi|(S_A)$, and the system of norms $\|\cdot\|_A$, $A \in]0,\infty[$ will then induce the Fréchét space topology on $\mathscr{B}$. For a set $\mathscr{M} \subseteq \mathscr{B}$ we can define a generalized diameter by $\mathrm{diam}_A \mathscr{M} = \sup\{\|\psi - \varphi\|_A \mid \varphi, \psi \in \mathscr{M}\}$. It is an increasing function of A and it may of course be infinite. For $\underline{x}, \underline{x}' \in \psi_n(X_n) \cap U_n(\underline{x}^\circ)$ we have $x, \tau \in \mathbf{R}$ and $\underline{\nu} = (\nu_1,\dots, \nu_n)$, $\underline{\nu}' = (\nu_1',\dots, \nu_n')$, $\nu_j, \nu_j' \in \mathbf{Z}, j = 1,\dots, n$ and $\underline{x} = \psi_n(x,\underline{\nu})$, $\underline{x}' = \psi_n(x + \tau,\underline{\nu}')$. The corresponding functions of $W_n(\underline{x}^\circ)$ are $g_{n,\underline{x}} = g_x$ and $g_{n,\underline{x}'} = g_{x+\tau}$. But x and $x+\tau$ satisfy (4) with $q = n$, hence, τ satisfies (1) with $\delta = \frac{1}{n}$, and Proposition 3 implies that $\mathrm{diam}_A W_n(\underline{x}^\circ) \to 0$ for $n \to \infty$ and fixed A, and uniformly for $\underline{x}^\circ \in \mathbf{R}^\infty$.

For $n, q \in \mathbf{N}$ we observe that those $\underline{x} = \psi_n(x,\underline{\nu})$ which have $\nu_1,\dots, \nu_n$ divisible by $(n+1) \dots (n + q)$ are also in $\psi_{n+q}(X_n)$ and it follows that some functions of $W_n(\underline{x}^\circ)$ are also in $W_{n+q}(\underline{x}^\circ)$. With $\tilde{W}_n(\underline{x}^\circ) = \bigcup_{q=0}^\infty W_{n+q}(\underline{x}^\circ)$ we can thus conclude that

$$\operatorname{diam}_A \tilde{W}_n(\underline{x}^\circ) \leqq 2 \operatorname{diam}_A W_n(\underline{x}^\circ).$$

It follows from this that every function of $W_n(\underline{x}^\circ)$ for $n \to \infty$ converges uniformly to the same limit function $g_{\underline{x}^\circ} : \mathbf{C} \to \mathbf{C}$, and it obviously is in T_g. The totality of functions $g_{\underline{x}}$ in every $C_{\underline{x}}$ constitutes a function $G : \mathbf{R}^\infty \times \mathbf{R} \to \mathbf{C}$. It follows immediately from the construction that G is continuous and that $G(\underline{\gamma}x;y) = g(z)$. The limit periodicity of G follows easily from the periodicity of G_n, since $\psi_n(X_n)$ is everywhere dense. This also implies that G is unique and that finishes the proof.

We could have derived it more easily from the approximation theorem, but the proof above underlines certain structural details, which are useful in our investigations.

The hypothetical function $f : \mathbf{C} \to \mathbf{C}$ with the basis $\underline{\gamma} = (\gamma_1, \gamma_1, \ldots)$ where $\gamma_1 = \dfrac{\beta_1}{\Delta}, \gamma_2 = \dfrac{\beta_2}{\Delta}$ has a spatial extension $F : \mathbf{R}^\infty \times \mathbf{R} \to \mathbf{C}$. We shall compute the zeros of F.

Lemma 5. *$F^{-1}(0)$ is the set E given by*

$$E = \{(\alpha_1 t + q_1, \alpha_2 t + q_2, x_3, x_4, \ldots; \Delta t) \mid t, x_3, x_4, \ldots \in \mathbf{R}, q_1, q_2 \in \mathbf{Z}\}.$$

Proof: We shall consider the set E defined in the lemma and we shall prove that it is identical to $F^{-1}(0)$. First, we determine $E \cap C_{\underline{x}^\circ}$ when $C_{\underline{x}^\circ} = \{(\underline{x}^\circ + \underline{\gamma}x; y) \mid x, y \in \mathbf{R}\}$ is an analytic plane. To do that we must determine the sets $(x, y, t; x_3, x_4, \ldots)$ satisfying the equations

$$\alpha_1 t + q_1 = x_1^\circ + \gamma_1 x,\ \alpha_2 t + q_2 = x_2^\circ + \gamma_2 x,\ \Delta t = y$$
$$x_j = x_j^\circ + \gamma_j x,\quad j = 3, 4, \ldots.$$

With $\gamma_1 = \dfrac{\beta_1}{\Delta}, \gamma_2 = \dfrac{\beta_2}{\Delta}$ the equations in the top row yield

$$x = x_1^\circ\alpha_2 - x_2^\circ\alpha_1 + q_2\alpha_1 - q_1\alpha_2; y = \Delta t = x_1^\circ\beta_2 - x_2^\circ\beta_1 + q_2\beta_1 - q_1\beta_2,$$

and $x_3, x_4, \ldots$ are determined by the equations in the second row. We are not really interested in these additional unknowns. We get

$$z = x + iy = x_1^\circ\omega_2 - x_2^\circ\omega_1 + q_2\omega_1 - q_1\omega_2.$$

We have thus proved that E intersects each analytic plane in $\mathbf{R}^\infty \times \mathbf{R}$ in a translated lattice spanned by ω_1 and ω_2.

It follows from Propositions 2 and 6 that also $F^{-1}(0)$ intersects each analytic plane in $\mathbf{R}^\infty \to \mathbf{R}$ in a translated lattice spanned by ω_1 and ω_2. To prove the theorem we need only that the two lattices in each analytic plane are identical, and that will follow when we have proved that

$$F(\alpha_1 t, \alpha_2 t, x_3, x_4, \ldots, \Delta t) = 0 \text{ for } t, x_3, x_4, \ldots \in \mathbf{R}.$$

By the limit periodicity of F it is enough to prove for every $\delta > 0$, $n \in \mathbf{N}$ that we can find $\omega \in \Omega$, $\omega = \alpha + i\beta$, such that

$$|\gamma_j \alpha - x_j| \leqq \delta \ (\mathrm{mod}\ n!\mathbf{Z}), j = 1, \ldots, n;\ x_1 = \alpha_1 t,\ x_2 = \alpha_2 t,\ |\beta - \Delta t| \leqq \delta. \tag{5}$$

We write $\omega = n_1\omega_1 + n_2\omega_2$, $n_1, n_2 \in \mathbf{Z}$, and (2) and (3) yield

$$\gamma_j \alpha - x_j = \gamma_j \frac{\Delta}{\beta_2} n_1 - x_j' + \varrho_j' = -\gamma_j \frac{\Delta}{\beta_1} n_2 - x_j'' + \varrho_j''$$

with

$$\begin{aligned} x_j' &= x_j - \gamma_j \frac{\alpha_2}{\beta_2} \Delta t, & \rho_j' &= \gamma_j \frac{\alpha_2}{\beta_2} (\beta - \Delta t), \\ x_j'' &= x_j - \gamma_j \frac{\alpha_1}{\beta_1} \Delta t, & \rho''_j &= \gamma_j \frac{\alpha_1}{\beta_1} (\beta - \Delta t), \end{aligned} \tag{6}$$

We introduce $\gamma = \max(|\gamma_j \frac{\alpha_k}{\beta_k}| \mid j = 1, \ldots, n;\ k = 1, 2)$ and $\delta' = \frac{\delta}{1+\gamma}$. Then $w = \alpha + i\beta = n_1\omega_1 + n_2\omega_2$ will satisfy (5) if n_1, n_2 satisfy first that $|n_1\beta_1 + n_2\beta_2 - \Delta t| \leqq \delta'$ and second for each $j \in \mathbf{N}$ one of the following Diophanthine inequalities

$$\left|\gamma_j \frac{\Delta}{\beta_2} n_1 - x_j'\right| \leqq \delta' \ (\mathrm{mod}\ n!\mathbf{Z}) \quad \text{or} \quad \left|-\gamma_j \frac{\Delta}{\beta_1} n_2 - x_n''\right| \leqq \delta' \ (\mathrm{mod}\ n!\mathbf{Z}).$$

For $j = 1, 2$ we insert $\gamma_j = \frac{\beta_j}{\Delta}$ and x_j', x_j'' from (6) and we get the inequalities

$$\begin{aligned} &\left|\frac{\beta_1}{\beta_2} n_1 - \frac{\Delta}{\beta_2} t\right| \leqq \delta' \ (\mathrm{mod}\ n!\mathbf{Z}) \quad \text{or} \quad |-n_2| \leqq \delta' \ (\mathrm{mod}\ n!\mathbf{Z}) \\ &|n_1| \leqq \delta' \ (\mathrm{mod}\ n!\mathbf{Z}) \quad \text{or} \quad \left|-\frac{\beta_2}{\beta_1} n_2 + \frac{\Delta}{\beta_1} t\right| \leqq \delta' \ (\mathrm{mod}\ n!\mathbf{Z}). \end{aligned} \tag{7}$$

The second and third of these are satisfied by $n_j = n!\nu_j$, $\nu_j \in \mathbf{Z}$, $j = 1, 2$. The inequality $|n_1\beta_1 + n_2\beta_2 - \Delta t| \leqq \delta'$ becomes

$$|n!\beta_1\nu_1 + n!\beta_2\nu_2 - \Delta t| \leqq \delta'.$$

We observe that the first and the fourth of the inequalities (7) follow from this last inequality and, further, that the last inequality is satisfied by some $\nu_2 \in \mathbf{Z}$, if ν_1 satisfies that $|n!\frac{\beta_1}{\beta_2}\nu_1 - \frac{\Delta}{\beta_2}t| \leqq \frac{\delta'}{|\beta_2|}$ (mod $n!\mathbf{Z}$). Hence, (5) will be satisfied, if $\nu_1 \in \mathbf{Z}$ can be chosen as a solution to the following system of Diophanthine inequalities:

$$\left| n!\frac{\beta_1}{\beta_2}\nu_1 - \frac{\Delta}{\beta_2}t \right| \leqq \frac{\delta'}{|\beta_2|}(\mathrm{mod}\ n!\mathbf{Z});$$

$$\left| \gamma_j\frac{\Delta}{\beta_2}n!\nu_1 - x_j' \right| \leqq \delta'\ (\mathrm{mod}\ n!\mathbf{Z}), \quad j = 3, 4, \ldots, n.$$

By a slightly advanced form of Kronecker's theorem we have that this system has integral solutions for all t, x_j', $j = 3, 4, \ldots$, if and only if no linear combination of the coefficients $n!\frac{\beta_1}{\beta_2}$, $n!\gamma_j\frac{\Delta}{\beta_2}$, $j = 3, 4, \ldots$ with integral coefficients has an integral value different from 0. In other words, solutions exist, if

$$q_2 + q_1\frac{\beta_1}{\beta_2} + q_3\gamma_3\frac{\Delta}{\beta_2} + \cdots + q_n\gamma_n\frac{\Delta}{\beta_2} = 0, \quad q_1, \ldots, q_n \in \mathbf{Z}$$

implies that $q_1 = \cdots = q_n = 0$. However, the equation can be written

$$q_1\frac{\beta_1}{\Delta} + q_2\frac{\beta_2}{\Delta} + q_3\gamma_3 + \cdots + q_n\gamma_n = 0,$$

and $\frac{\beta_1}{\Delta}, \frac{\beta_2}{\Delta}\ \gamma_3, \gamma_4, \ldots$ are independent. That proves the lemma.

LEMMA 6. *Let* $\mathbf{R}^2 \times \mathbf{R} \subset \mathbf{R}^\infty \times \mathbf{R}$ *be the* $(x_1, x_2; y)$*-subspace, and* $p : \mathbf{R}^\infty \times \mathbf{R} \to \mathbf{R}^2 \times \mathbf{R}$ *the projection. Then* $E_0 = E \cap (\mathbf{R}^2 \times \mathbf{R})$ *is a system of parallel straight lines and* $F^{-1}(0) = E = p^{-1}(E_0)$. *Further* E_0 *contains exactly one straight line* L_{q_1,q_2} *through each point* $(q_1, q_2, 0)$ *of the unit lattice in* $\mathbf{R}^2$. *The sets* $p^{-1}(L_{q_1,q_2})$ *are the components of* E *and* L_{q_1,q_2} *intersects the analytic plane* C *in the point corresponding to* $z = q_2\omega_1 - q_1\omega_2$.

This is nothing more than a reformulation of Lemma 5 supplemented by very few and very elementary computations.

The non existence of f

The spatial extension F of f has a restriction $\varphi : \mathbf{R}^2 \to \mathbf{C}$ defined by $\varphi(x_1, x_2) = F(x_1, x_2, 0, 0, \ldots, 0)$. We know that φ^{-1} (0) is the unit lattice in $\mathbf{R}^2$. The midway net $M \subset \mathbf{R}^2$ is defined as the set of all points (x_1, x_2) with either x_1 or x_2 equal to $\frac{1}{2}$ + some integer. It divides $\mathbf{R}^2$ in unit squares such that φ has one zero in the center of each square.

Lemma 7. $\inf |\varphi|(M) = k > 0$.

Proof: In each analytic plane $C_{\underline{x}} = \{(\underline{x} + \underline{\gamma} x; y) | x, y \in \mathbf{R}\}$ we place discs defined by $x = \alpha + \rho \cos \theta, y = \rho \sin \theta$; $\theta \in \mathbf{R}$, $\varrho \in]0,r[$ for some $r \in]0,r_0[$ (Lemma 2) and for each $\omega = \alpha + i\beta \in \Omega$. The union of all these discs is by Lemma 5 the set of all points of $\mathbf{R}^\infty \times \mathbf{R}$ given by

$$\begin{aligned}&(\alpha_1 t + q_1 + \gamma_1 \rho \cos \theta,\ \alpha_2 t + q_2 + \gamma_2 \rho \cos \theta,\\ &x_3 + \gamma_3 \rho \cos \theta,\ x_4 + \gamma_4 \rho \cos \theta, \ldots;\ \Delta t + \rho \sin \theta)\\ &\theta, t, x_3, x_4, \cdots \in \mathbf{R}, \quad \rho \in [0,r], \quad q_1, q_2 \in \mathbf{Z}.\end{aligned}$$

Let us denote this set E_r and its intersection with $\mathbf{R}^2 \times \mathbf{R}$ by E_r^0. It follows immediately from the expression or from Lemma 6 that $\underline{E}_r = p^{-1}(E_r^0)$ and we have

$$E_r^0 = \left\{ \left(\alpha_1 t + q_1 + \frac{\beta_1 \rho}{\Delta} \cos \theta,\ \alpha_2 t + q_2 + \frac{\beta_2 \rho}{\Delta} \cos \theta;\ \Delta t + \rho \sin \theta \right) \ \middle|\ t, \theta \in \mathbf{R}, \rho \in [0, r], q_1, q_2 \in \mathbf{Z} \right\}.$$

The intersection of E_r^0 with the (x_1, x_2)-plane is

$$\tilde{E}_r = \{(q_1 + \frac{\varrho}{\Delta}(\beta_1 \cos \theta - \alpha_1 \sin \theta),$$

$$q_2 + \frac{\rho}{\Delta}(\beta_2 \cos \theta - \alpha_2 \sin \theta)) \mid \theta \in \mathbf{R}, \rho \in [0,r], q_1, q_2 \in \mathbf{Z}\}.$$

This set consists of elliptic discs with centers in each point of the unit lattice and they are exactly alike and oriented in the same manner. We choose a fixed value of r such that $\tilde{E}_r \cap M = \emptyset$.

We know from Proposition 6 that the restriction of F to an arbitrary analytic plane is a function of T_f. Hence, the lemma follows from Lemma 3 with $A > 0$ chosen arbitrarily. This ends the proof.

Lemma 8. *There is an orientation of the (x_1, x_2)-plane such that the variation of the argument of φ along a small circle around a lattice point and in the direction given by the orientation of the plane is equal to 2π for every point of the unit lattice.*

Proof: For $u \in [0,1]$ and $\underline{\gamma}_u = (\gamma_1, \gamma_2, u\gamma_3, u\gamma_4, \ldots)$ we have the family of planes $C_u = \{\underline{\gamma}_u x; y) \mid x, y \in \mathbf{R}\}$ in $\mathbf{R}^\infty \times \mathbf{R}$ and for each $\omega = \alpha + i\beta \in \Omega$ and $r \in]0, r_0[$ we get a family of circles

$$\Gamma_u = \{(\underline{\gamma}_u(\alpha + r \cos \theta); \beta + r \sin \theta) \mid \theta \in \mathbf{R}\}.$$

From Lemma 6 follows that Γ_u is a continuous family of circles in $\mathbf{R}^\infty \times \mathbf{R} \setminus F^{-1}(0)$. We choose the orientation of each plane C_u such that the angle from the x-axis to the y-axis is $+\frac{\pi}{2}$. Then, the variation of the argument of F along Γ_u has its value independent of u, and since the restriction of F to C_1 is an entire function we conclude that F has its variation of argument equal to 2π along $\Gamma_0 \subset C_0$ for every $\omega \in \Omega$.

From now on we shall consider only the restriction $\tilde{F} : \mathbf{R}^2 \times \mathbf{R} \to \mathbf{C}$ of F defined by $\tilde{F}(x_1, x_2; y) = F(x_1, x_2, 0, 0, \ldots; y)$. We have $C_0 \subset \mathbf{R}^2 \times \mathbf{R}$. It will be convenient to think of $\mathbf{R}^2 \times \mathbf{R}$ as our physical space with the y-axis vertical, and C_0 is then raised as a vertical wall, which divides $\mathbf{R}^2 \times \mathbf{R}$ in two half spaces V_d and V_u such that the lines $L_{q_1 q_2}$ slant downwards in V_d and upwards in V_u. The set E_r^0 from the proof of Lemma 7 is the union of disjoint elliptic cylinders such that each $L_{q_1 q_2}$ is the axis of symmetry of one of them. The circles Γ_0 induce an orientation of each cylinder and we know that the variation of the argument of $\tilde{F}$ along a curve encircling a cylinder once is 2π. In particular this holds for the ellipses, in which E_r^0 intersects the (x_1, x_2)-plane. The orientation of the (x_1, x_2)-plane corresponding to this can be determined in the following way: Start with a circle $\Gamma \subset C_0$ with a diameter in the (x_1, x_2)-plane and oriented according to C_0. Rotate it an angle $\frac{\pi}{2}$ about the horizontal diameter such that its upper half goes into V_d, and it yields the orientation. This finishes the proof.

It must be obvious to everybody that the lemmas 7 and 8 contradict each other, but we must go through the details anyway such that our proof is not left unfinished.

Theorem 1. *Let $\Omega \subset \mathbf{C}$ be a lattice with no real period. Then no entire function $f : \mathbf{C} \to \mathbf{C}$ almost periodic in the direction of the real axis will satisfy that $f^{-1}(0) = \Omega$.*

Proof: If the theorem was false, our hypothetical function f would exist and the lemmas 7 and 8 would hold for some limit periodic function $\varphi : \mathbf{R}^2 \to \mathbf{C}$. Let Γ be the oriented boundary of a square on the midway net M and the length of the sides $n!$ for some large $n \in \mathbf{N}$. Let $v \in \mathbf{R}$ be the variation of the argument of φ along Γ.

By Lemma 8 and the ordinary routine we get $v = 2\pi(n!)^2$.

We choose n large enough such that for every $(x_1, x_2) \in \mathbf{R}^2$

$$|\varphi(x_1, x_2 + n!) - \varphi(x_1, x_2)| \leqq \frac{1}{2} k, \quad |\varphi(x_1 + n!, x_2) - \varphi(x_1, x_2)| \leqq \frac{1}{2} k.$$

Then the variations of the argument of φ along the sides of Γ parallel to the x_1-axis taken together amounts to the variation of the argument of $\frac{\varphi(x_1, x_2 + n!)}{\varphi(x_1, x_2)}$, but this quotient is contained in the angle defined by $|\arg z| \leqq \frac{\pi}{6}$, hence the variation of the argument along these sides amount to at most $\frac{\pi}{3}$. The same holds for the two other sides, and since the variation of the argument along Γ is an integer multiplied by 2π, we can conclude that $v = 0$.

This proves the theorem.

Application of Weierstrass' σ-function

We shall use the notations Ω, ω_1, ω_2, α_1, β_1, β_2, Δ as before. With $\Omega' = \Omega \setminus \{0\}$ Weierstrass' σ-function is defined by

$$\sigma(z) = z\Pi_{\omega \epsilon \Omega'} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{z^2}{2\omega^2}}.$$

It is an entire function of order 2 with $\sigma^{-1}(0) = \Omega$, and though it is not periodic, there are constants $\eta_1, \eta_2 \epsilon \mathbf{C}$ satisfying

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i \tag{8}$$

such that σ has the periodicity property

$$\sigma(z + \omega_j) = -e^{\eta_j\left(z+\frac{1}{2}\omega_j\right)}\sigma(z), \quad j = 1, 2$$

and for $\omega = n_1\omega_1 + n_2\omega_2$ with $\eta = n_1\eta_1 + n_2\eta_2$ we have generally

$$\sigma(z + \omega) = (-1)^{n_1n_2+n_1+n_2}e^{\eta\left(z+\frac{1}{2}\omega\right)}\sigma(z).$$

From this follows that the function $f_\omega : \mathbf{C} \to \mathbf{C}$ defined by

$$f_\omega(z) = e^{-\frac{\eta}{2\omega}z^2}\sigma(z)$$

has period 2ω and even ω if n_1 and n_2 are even. We remark that f_ω depends only on the direction of ω not on ω itself. We supply (3) with a corresponding formula for η so that we have

$$\omega = -n_2\frac{\Delta}{\beta_1} + \frac{\omega_1}{\beta_1}\beta, \quad \eta = -n_2\frac{\eta_1\beta_2 - \eta_2\beta_1}{\beta_1} + \frac{\eta_1}{\beta_1}\beta,$$

and if we let $\omega \to \infty$ while $\beta \to 0$, the ratio $\frac{\eta}{\omega}$ will tend to

$$\gamma = \frac{\eta_1\beta_2 - \eta_2\beta_1}{\Delta}$$

and $f_\omega(z)$ tends to the limit

$$f(z) = e^{-\frac{\gamma}{2}z^2}\,\sigma(z),$$

an entire function with $f^{-1}(0) = \Omega$ and obviously satisfying

$$f(z+\omega) = \pm\, e^{(\eta-\gamma\omega)\,(z+\frac{1}{2}\omega)}\, f(z).$$

We do some computation

$$\begin{aligned}\eta - \gamma\omega &= \frac{1}{\Delta\beta_1}(-\eta_2\Delta(\eta_1\beta_2 - \eta_2\beta_1) + \Delta\eta_1\beta - (\eta_1\beta_2 - \eta_2\beta_1)\,(-\eta_2\Delta + \omega_1\beta))\\ &= \frac{1}{\Delta\beta_1}((\alpha_1\beta_2 - \alpha_2\beta_1)\eta_1\beta - (\eta_1\beta_2 - \eta_2\beta_1)\omega_1\beta),\end{aligned}$$

but since

$$\alpha_1\beta_2 - \alpha_2\beta_1 = (\omega_1 - i\beta_1)\beta_2 - (\omega_2 - i\beta_2)\beta_1 = \omega_1\beta_2 - \omega_2\beta_1,$$

it reduces to

$$\eta - \gamma\omega = \frac{1}{\Delta}(-\eta_1\omega_2 + \eta_2\omega_1)\beta = -2\pi i\,\frac{\beta}{\Delta}$$

by (8). Thus we have

$$f(z+\omega) = \pm\, e^{-2\pi\frac{\beta}{\Delta}\,(z+\frac{1}{2}\omega)} f(z)$$

and with $\omega = \alpha + i\beta$, $z = iy$ this implies

$$|f(z+\omega)| = e^{-2\pi\frac{\beta}{\Delta}\,(y+\frac{1}{2}\beta)}\,|f(z)|.$$

We know from Theorem 1 that $f : \mathbf{C} \to \mathbf{C}$ is not almost periodic. Nevertheless, we have the following

THEOREM 2. *The function* $|f| : \mathbf{C} \to \mathbf{R}$ *is almost periodic.*

Proof: We shall first prove that f is bounded in every strip, hence, we consider S_A, $A > 0$. We choose $L > 0$ such that every interval on $\mathbf{R}$ of length L contains a number α for which there is a $\beta \in [-1, 1]$ with $\alpha + i\beta = \omega \in \Omega$. We define $K = \max|f|([0, L] \times [-A-1, A+1])$ and for an arbitrary $z \in S_A$ we can then find $\omega = \alpha + i\beta \in \Omega$ with $|\beta| \leqq 1$ and $x - \alpha \in [0, L]$, hence $z - \omega \in [0, L] \times [-A-1, A+1]$. It follows that

$$|f(z)| \leqq K\, e^{2\pi \frac{|\beta|}{\Delta}(A+1)},$$

and this proves that f is bounded in S_A, i.e. in every strip. Since f is entire, this implies that f is uniformly continuous in every strip.

Let $\varepsilon > 0$ be given. We choose $\delta_1 \in]0, 1]$ such that for every $z \in S_A$ and every $\omega = \alpha + i\beta$ with $|\beta| \leqq \delta$, we have

$$|f(z+\omega) - f(z+\alpha)| \leqq \frac{1}{2}\varepsilon.$$

With the K introduced above we choose $\delta_2 > 0$ such that for $|\beta| < \delta_2$

$$\left| e^{2\pi\frac{\beta}{A}(A+\frac{1}{2}|\beta|)} - 1 \right| \leqq \frac{\varepsilon}{2K}\, e^{-\frac{2\pi}{A}(A+\frac{3}{2})},$$

and with $\delta = \min(\delta_1, \delta_2)$ for $|\beta| \leqq \delta$ we have

$$\big||f(z+\omega)| - |f(z)|\big| \leqq \left| e^{2\pi\frac{\beta}{\Delta}(A+\frac{1}{2}|\beta|)} - 1 \right| |f(z)| \leqq \frac{1}{2}\varepsilon,$$

and together with the preceding inequality this proves that α is an (ε, A)-translation number of $|f|$, and that proves the theorem.

Theorem 3. *With $\overline{\Omega} = \{\omega \in \mathbf{C} \mid \overline{\omega} \in \Omega\}$ there is an entire almost periodic function $g : \mathbf{C} \to \mathbf{C}$ of order 2 and with $g^{-1}(0) = \Omega \cup \overline{\Omega}$ such that the elements of $\Omega \cup \overline{\Omega}$ are simple zeros, except 0 which is double.*

Proof: We define g by $g(z) = f(z)\overline{f(\bar{z})}$ and $g : \mathbf{C} \to \mathbf{C}$ is entire of order 2 and $g^{-1}(0)$ is as claimed in the theorem. By the multiplication theorem $|g| : \mathbf{C} \to \mathbf{R}$ is almost periodic, hence g is bounded in every strip. For $z = x \in \mathbf{R}$ we have $g(x) = |g|(x)$, hence, g is almost periodic on the real axis. But this implies that g is almost periodic in every strip ([2], p. 253). This proves the theorem.

A more general entire almost periodic function with $g^{-1}(0) \supset \Omega$ could be defined by $g_a(z) = f(z+a)\,\overline{f(\bar{z}+\bar{a})}$ for some $a \in \mathbf{C}$.

Bibliography

The few facts about elliptic functions used in the paper can be found in most comprehensive texts. The author recommends the presentation in Hurwitz-Courant: Funktionentheorie, because it introduces the relevant functions at an early stage.

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